

# FINITE ENERGY SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS WITH SUB-NATURAL GROWTH TERMS

CAO TIEN DAT AND IGOR E. VERBITSKY

ABSTRACT. We study finite energy solutions to quasilinear elliptic equations of the type

$$-\Delta_p u = \sigma u^q \quad \text{in } \mathbb{R}^n,$$

where  $\Delta_p$  is the  $p$ -Laplacian,  $p > 1$ , and  $\sigma$  is a nonnegative function (or measure) on  $\mathbb{R}^n$ , in the case  $0 < q < p - 1$  (below the “natural growth” rate  $q = p - 1$ ). We give an explicit necessary and sufficient condition on  $\sigma$  which ensures that there exists a solution  $u$  in the homogeneous Sobolev space  $L_0^{1,p}(\mathbb{R}^n)$ , and prove its uniqueness. Among our main tools are integral inequalities closely associated with this problem, and Wolff potential estimates used to obtain sharp bounds of solutions. More general quasilinear equations with the  $\mathcal{A}$ -Laplacian  $\operatorname{div} \mathcal{A}(x, \nabla \cdot)$  in place of  $\Delta_p$  are considered as well.

## 1. INTRODUCTION

This paper is concerned with quasilinear problems of the following type:

$$(1.1) \quad -\Delta_p u = \sigma u^q \quad \text{in } \mathbb{R}^n,$$

where  $\Delta_p u = \nabla \cdot (\nabla u |\nabla u|^{p-2})$  is the  $p$ -Laplacian,  $1 < p < \infty$ , and  $\sigma$  is a nonnegative function, or measure, in the *sub-natural growth* case  $0 < q < p - 1$ . We are interested in finite energy solutions  $u \in L_0^{1,p}(\mathbb{R}^n)$  to (1.1), and related integral inequalities. Here  $L_0^{1,p}(\mathbb{R}^n)$  is the homogeneous Sobolev (or Dirichlet) space defined in Sec. 2 (see [HKM06], [MZ97], [Maz11]); for  $1 < p < n$  it can be identified with the completion of  $C_0^\infty(\mathbb{R}^n)$  in the norm

$$(1.2) \quad \|u\|_{1,p} = \left( \int_{\mathbb{R}^n} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}.$$

More precisely,  $u$  is called a finite energy solution to (1.1) if  $u \in L_0^{1,p}(\mathbb{R}^n) \cap L_{\text{loc}}^q(\mathbb{R}^n, d\sigma)$ ,  $u \geq 0$ , and, for all  $\varphi \in C_0^\infty(\mathbb{R}^n)$ ,

$$(1.3) \quad \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_{\mathbb{R}^n} u^q \varphi d\sigma.$$

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Finite energy solutions to (1.1) are critical points of the functional

$$H[\varphi] = \int_{\mathbb{R}^n} \frac{1}{p} |\nabla \varphi|^p dx - \int_{\mathbb{R}^n} \frac{1}{q+1} |\varphi|^{1+q} d\sigma.$$

We will give a *necessary and sufficient* condition for the existence of a finite energy solution to (1.1), and prove its uniqueness.

Our results are new even in the classical case  $p = 2$ ,  $0 < q < 1$ . Sublinear elliptic problems of this type were studied by Brezis and Kamin in [BrK92], where a necessary and sufficient condition is found for the existence of a *bounded* solution on  $\mathbb{R}^n$ , together with sharp pointwise estimates of solutions. Recently, we have extended these results to the case  $p \neq 2$ , under relaxed assumptions on  $\sigma$ , in such a way that some singular (unbounded) solutions are covered as well [CV13]. However, the techniques used in [CV13] are quite different from those used in this paper.

Analogous sublinear problems in bounded domains  $\Omega \subset \mathbb{R}^n$  for various classes of  $\sigma$  have been extensively studied. In particular, Boccardo and Orsina [BO96], [BO12], and Abdel Hamid and Bidaut-Véron [ABV10] gave sufficient conditions for the existence of solutions under the assumption  $\sigma \in L^r(\Omega)$ . Earlier results, under more restrictive assumptions on  $\sigma$ , can be found in Krasnoselskii [Kr64], Brezis and Oswald [BrO86], and the literature cited in these papers.

We employ powerful Wolff potential estimates developed in [KM94] (see also [Lab02], [TW02], [KuMi13]). This makes it possible to replace the  $p$ -Laplacian  $\Delta_p$  in the model problem (1.1) by a more general quasilinear operator  $\operatorname{div} \mathcal{A}(x, \nabla \cdot)$  with bounded measurable coefficients, under standard structural assumptions on  $\mathcal{A}(x, \xi)$  which ensure that  $\mathcal{A}(x, \xi) \cdot \xi \approx |\xi|^p$  [HKM06], [MZ97], or a fully nonlinear operator of  $k$ -Hessian type [TW99], [Lab02] (see also [PV09], [JV12]), and treat more general nonlinearities on the right-hand side. Equations involving operators of the  $p$ -Laplacian type on Carnot groups can be covered as well using methods developed in [PV13].

Wolff's potential  $\mathbf{W}_{1,p}\sigma$  of a nonnegative Borel measure  $\sigma$  on  $\mathbb{R}^n$  is defined by [HW83] (see also [AH96]):

$$(1.4) \quad \mathbf{W}_{1,p}\sigma(x) = \int_0^\infty \left( \frac{\sigma(B(x,t))}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Here  $B(x, t) = \{y \in \mathbb{R}^n : |x - y| < t\}$  is a ball centered at  $x \in \mathbb{R}^n$  of radius  $t > 0$ .

An important theorem due to Kilpeläinen and Malý [KM94] (see also [Kil03]) states that if  $U$  is a solution (understood in the potential theoretic or renormalized sense) to the equation

$$(1.5) \quad \begin{cases} -\Delta_p U = \sigma & \text{in } \mathbb{R}^n, \\ \inf_{\mathbb{R}^n} U = 0, \end{cases}$$

then there exists a constant  $K > 0$  which depends only on  $p$  and  $n$  such that

$$(1.6) \quad \frac{1}{K} \mathbf{W}_{1,p}\sigma(x) \leq U(x) \leq K \mathbf{W}_{1,p}\sigma(x), \quad x \in \mathbb{R}^n.$$

Moreover,  $U$  exists if and only if  $\mathbf{W}_{1,p}\sigma \not\equiv +\infty$  (see [PV08]), or equivalently,

$$(1.7) \quad \int_1^\infty \left( \frac{\sigma(B(0,t))}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} < +\infty.$$

Our main result is the following

**Theorem.** *Let  $0 < q < p-1$ ,  $1 < p < n$ , and let  $\sigma$  be a locally finite positive measure on  $\mathbb{R}^n$ . Then there exists a nontrivial solution  $u \in L_0^{1,p}(\mathbb{R}^n) \cap L_{loc}^q(\Omega, d\sigma)$  to (1.1) if and only if  $U \in L^{\frac{(1+q)(p-1)}{p-1-q}}(\mathbb{R}^n, d\sigma)$ , or equivalently,*

$$(1.8) \quad \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\sigma)^{\frac{(1+q)(p-1)}{p-1-q}} d\sigma < \infty.$$

Furthermore, such a solution is unique. For  $p \geq n$ , (1.1) has only a trivial solution  $u = 0$ .

We observe that (1.8) yields  $\sigma \in L_{loc}^{-1,p'}(\mathbb{R}^n)$ , where  $L^{-1,p'}(\mathbb{R}^n) = L_0^{1,p}(\mathbb{R}^n)^*$  is the dual Sobolev space (see definitions in Sec. 2). Consequently,  $\sigma$  is necessarily absolutely continuous with respect to the  $p$ -capacity  $\text{cap}_p(\cdot)$  defined by

$$(1.9) \quad \text{cap}_p(E) = \inf\{ \|\nabla \phi\|_{L^p}^p : \phi \geq 1 \text{ on } E, \phi \in C_0^\infty(\mathbb{R}^n) \},$$

for a compact set  $E \subset \mathbb{R}^n$ .

Moreover, as was shown in [COV00], condition (1.8) holds if and only if there exists a constant  $C$  such that, for all  $\varphi \in C_0^\infty(\mathbb{R}^n)$ ,

$$(1.10) \quad \left( \int_{\mathbb{R}^n} |\varphi|^{1+q} d\sigma \right)^{\frac{1}{1+q}} \leq C \|\nabla \varphi\|_{L^p(\mathbb{R}^n)}.$$

An obvious sufficient condition which follows from Sobolev's inequality is  $\sigma \in L^r(\mathbb{R}^n)$ ,  $r = \frac{np}{n(p-1-q)+p(1+q)}$ .

There is also an equivalent characterization of (1.10) in terms of capacities due to Maz'ya and Netrusov (see [Maz11], Sec. 11.6):

$$(1.11) \quad \int_0^{\sigma(\mathbb{R}^n)} \left[ \frac{t}{\varkappa(\sigma, t)} \right]^{\frac{1+q}{p-1-q}} dt < +\infty,$$

where  $\varkappa(\sigma, t) = \inf\{ \text{cap}_p(E) : \sigma(E) \geq t \}$ .

Thus, any one of the conditions (1.8), (1.10), and (1.11) is necessary and sufficient for the existence of a nontrivial finite energy solution to (1.1).

We now outline the contents of the paper. Sec. 2 contains definitions and notations, along with several useful results on quasilinear equations that will be used below. In Sec. 3 we study the corresponding integral inequalities, deduce a necessary and sufficient condition for the existence of a finite energy solution, and construct a minimal solution. Sec. 4 is devoted to more general

equations with the operator  $\operatorname{div} \mathcal{A}(x, \nabla \cdot)$  in place of the  $p$ -Laplacian. In Sec. 5 we prove the uniqueness property of finite energy solutions.

## 2. PRELIMINARIES

We first recall some notations and definitions. Given an open set  $\Omega \subseteq \mathbb{R}^n$ , we denote by  $M^+(\Omega)$  the class of all nonnegative Borel measures in  $\Omega$  which are finite on compact subsets of  $\Omega$ . The  $\sigma$ -measure of a measurable set  $E \subset \Omega$  is denoted by  $|E|_\sigma = \sigma(E) = \int_E d\sigma$ .

For  $p > 0$  and  $\sigma \in M^+(\Omega)$ , we denote by  $L^p(\Omega, d\sigma)$  ( $L_{\text{loc}}^p(\Omega, d\sigma)$ , respectively) the space of measurable functions  $\varphi$  such that  $|\varphi|^p$  is integrable (locally integrable) with respect to  $\sigma$ . For  $u \in L^p(\Omega, d\sigma)$ , we set

$$\|u\|_{L^p(\Omega, d\sigma)} = \left( \int_\Omega |u|^p d\sigma \right)^{\frac{1}{p}}.$$

When  $d\sigma = dx$ , we write  $L^p(\Omega)$  (respectively  $L_{\text{loc}}^p(\Omega)$ ), and denote Lebesgue measure of  $E \subset \mathbb{R}^n$  by  $|E|$ .

The Sobolev space  $W^{1,p}(\Omega)$  ( $W_{\text{loc}}^{1,p}(\Omega)$ , respectively) is the space of all functions  $u$  such that  $u \in L^p(\Omega)$  and  $|\nabla u| \in L^p(\Omega)$  ( $u \in L_{\text{loc}}^p(\Omega)$  and  $|\nabla u| \in L_{\text{loc}}^p(\Omega)$ , respectively). By  $L_0^{1,p}(\Omega)$  we denote the homogeneous Sobolev space, i.e., the space of functions  $u \in W_{\text{loc}}^{1,p}(\Omega)$  such that  $|\nabla u| \in L^p(\Omega)$ , and  $\|\nabla u - \nabla \varphi_j\|_{L^p(\Omega)} \rightarrow 0$  as  $j \rightarrow \infty$  for a sequence  $\varphi_j \in C_0^\infty(\Omega)$ .

When  $1 < p < n$  and  $\Omega = \mathbb{R}^n$ , we will identify  $L_0^{1,p}(\mathbb{R}^n)$  with the space of all functions  $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$  such that  $u \in L^{\frac{np}{n-p}}(\mathbb{R}^n)$  and  $|\nabla u| \in L^p(\mathbb{R}^n)$ . For  $u \in L_0^{1,p}(\mathbb{R}^n)$ , the norm  $\|u\|_{1,p}$  is equivalent to

$$\|u\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} + \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

It is easy to see that  $C_0^\infty(\mathbb{R}^n)$  is dense in  $L_0^{1,p}(\mathbb{R}^n)$  with respect to this norm (see, e.g., [MZ97], Sec. 1.3.4).

If  $1 < p < n$  and  $\Omega = \mathbb{R}^n$ , then the dual Sobolev space  $L^{-1,p'}(\mathbb{R}^n) = L_0^{1,p}(\mathbb{R}^n)^*$  is the space of distributions  $\nu$  such that

$$\|\nu\|_{-1,p'} = \sup \frac{|\langle u, \nu \rangle|}{\|u\|_{1,p}} < +\infty,$$

where the supremum is taken over all  $u \in L_0^{1,p}(\mathbb{R}^n)$ ,  $u \neq 0$ . We write  $\nu \in L_{\text{loc}}^{-1,p'}(\mathbb{R}^n)$  if  $\varphi \nu \in L^{-1,p'}(\mathbb{R}^n)$ , for every  $\varphi \in C_0^\infty(\mathbb{R}^n)$ .

For  $u \in W_{\text{loc}}^{1,p}(\Omega)$ , we define the  $p$ -Laplacian  $\Delta_p$  ( $1 < p < \infty$ ), in the distributional sense, i.e., for every  $\varphi \in C_0^\infty(\Omega)$ ,

$$(2.1) \quad \langle \Delta_p u, \varphi \rangle = \langle \operatorname{div}(|\nabla u|^{p-2} \nabla u, \varphi) = - \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx.$$

A *finite energy* solution  $u \geq 0$  to (1.1) is understood in the sense that  $u \in L_0^{1,p}(\Omega) \cap L_{\text{loc}}^q(\Omega, d\sigma)$ , and, for every  $\varphi \in C_0^\infty(\Omega)$ ,

$$(2.2) \quad \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} u^q \varphi \, d\sigma.$$

We need to extend the definition of solutions to  $u$  not necessarily in  $W_{\text{loc}}^{1,p}(\Omega)$ . We will understand solutions in the following potential-theoretic sense using  $p$ -superharmonic functions, which is equivalent to the notion of locally renormalized solutions in terms of test functions (see [KKT09]).

A function  $u \in W_{\text{loc}}^{1,p}(\Omega)$  is called  $p$ -harmonic if it satisfies the homogeneous equation  $\Delta_p u = 0$ . Every  $p$ -harmonic function has a continuous representative which coincides with  $u$  a.e. (see [HKM06]).

As usual,  $p$ -superharmonic functions are defined via a comparison principle. We say that  $u: \Omega \rightarrow (-\infty, \infty]$  is  $p$ -superharmonic if  $u$  is lower semi-continuous, is not identically infinite in any component of  $\Omega$ , and satisfies the following comparison principle: Whenever  $D \subset\subset \Omega$  and  $h \in C(\bar{D})$  is  $p$ -harmonic in  $D$ , with  $h \leq u$  on  $\partial D$ , then  $h \leq u$  in  $D$ .

A  $p$ -superharmonic function  $u$  does not necessarily belong to  $W_{\text{loc}}^{1,p}(\Omega)$ , but its truncates  $T_k(u) = \min(u, k)$  do, for all  $k > 0$ . In addition,  $T_k(u)$  are supersolutions, i.e.,  $-\text{div}(|\nabla T_k(u)|^{p-2} \nabla T_k(u)) \geq 0$ , in the distributional sense. We will need the generalized gradient of a  $p$ -superharmonic function  $u$  defined by [HKM06]:

$$Du = \lim_{k \rightarrow \infty} \nabla(T_k(u)).$$

We note that every  $p$ -superharmonic function  $u$  has a quasicontinuous representative which coincides with  $u$  quasieverywhere (q.e.), i.e., everywhere except for a set of  $p$ -capacity zero. We will assume that  $u$  is always chosen this way.

Let  $u$  be  $p$ -superharmonic, and let  $1 \leq r < \frac{n}{n-1}$ . Then  $|Du|^{p-1}$ , and hence  $|Du|^{p-2} Du$ , belong to  $L_{\text{loc}}^r(\Omega)$  [KM92]. This allows us to define a nonnegative distribution  $-\Delta_p u$  for each  $p$ -superharmonic function  $u$  by

$$(2.3) \quad -\langle \Delta_p u, \varphi \rangle = \int_{\Omega} |Du|^{p-2} Du \cdot \nabla \varphi \, dx,$$

for all  $\varphi \in C_0^\infty(\Omega)$ . Then by the Riesz representation theorem there exists a unique measure  $\mu[u] \in M^+(\Omega)$  so that  $-\Delta_p u = \mu[u]$ .

**Definition 2.1.** *For a nonnegative locally finite measure  $\omega$  in  $\Omega$  we will say that*

$$-\Delta_p u = \omega \quad \text{in } \Omega$$

*in the potential-theoretic sense if  $u$  is  $p$ -superharmonic in  $\Omega$ , and  $\mu[u] = \omega$ .*

*Thus,  $-\Delta_p u = \sigma u^q$  if  $u \geq 0$  is  $p$ -superharmonic in  $\Omega$ ,  $u \in L_{\text{loc}}^q(\Omega, d\sigma)$ , and  $d\mu[u] = u^q d\sigma$ .*

**Definition 2.2.** A function  $u \geq 0$  is a supersolution to (1.1) if  $u$  is  $p$ -superharmonic,  $u \in L_{loc}^q(\Omega, d\sigma)$ , and, for every nonnegative  $\varphi \in C_0^\infty(\Omega)$ ,

$$(2.4) \quad \int_{\Omega} |Du|^{p-2} Du \cdot \nabla \varphi \, dx \geq \int_{\Omega} u^q \varphi \, d\sigma.$$

Supersolutions to (1.1) in the sense of Definition 2.2 are closely related to supersolutions associated with the integral equation

$$(2.5) \quad u = \mathbf{W}_{1,p}(u^q \, d\sigma) \quad d\sigma\text{-a.e.},$$

that is, measurable functions  $u \geq 0$  such that  $\mathbf{W}_{1,p}(u^q \, d\sigma) \leq u < \infty$   $d\sigma$ -a.e.

We will use the following universal lower bound for supersolutions obtained in [CV13].

**Theorem 2.3.** Let  $1 < p < n$ ,  $0 < q < p - 1$ , and  $\sigma \in M^+(\mathbb{R}^n)$ . Suppose  $u$  is a nontrivial  $p$ -superharmonic supersolution to (1.1). Then the inequality

$$(2.6) \quad u \geq C (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}$$

holds, where  $C$  is a positive constant depending only on  $p, q$ , and  $n$ .

The same lower bound holds for a nontrivial supersolution to the integral equation (2.5). If  $p \geq n$ , there is only a trivial supersolution  $u = 0$  on  $\mathbb{R}^n$ .

We will employ some fundamental results of the potential theory of quasilinear elliptic equations. The following important weak continuity result [TW02] will be used to prove the existence of  $p$ -superharmonic solutions to quasilinear equations.

**Theorem 2.4.** Suppose  $\{u_n\}$  is a sequence of nonnegative  $p$ -superharmonic functions that converges a.e. to a  $p$ -superharmonic function  $u$  in an open set  $\Omega$ . Then  $\mu[u_n]$  converges weakly to  $\mu[u]$ , i.e., for all  $\varphi \in C_0^\infty(\Omega)$ ,

$$\lim_{n \rightarrow \infty} \int_{\Omega} \varphi \, d\mu[u_n] = \int_{\Omega} \varphi \, d\mu[u].$$

The next result [KM94] is concerned with global pointwise estimates of nonnegative  $p$ -superharmonic functions in terms of Wolff's potentials discussed in the Introduction.

**Theorem 2.5.** Let  $1 < p \leq n$ . Let  $u$  be a  $p$ -superharmonic function in  $\mathbb{R}^n$  with  $\inf_{\mathbb{R}^n} u = 0$ . If  $\omega$  is a nonnegative Borel measure in  $\mathbb{R}^n$  such that  $-\Delta_p u = \omega$ , then

$$\frac{1}{K} \mathbf{W}_{1,p}\omega(x) \leq u(x) \leq K \mathbf{W}_{1,p}\omega(x), \quad x \in \mathbb{R}^n,$$

where  $K$  is a positive constant depending only on  $n, p$ .

The following theorem is due to Brezis and Browder [BrB79] (see also [MZ97], Theorem 2.39).

**Theorem 2.6.** *Let  $1 < p < n$ . Suppose  $u \in L_0^{1,p}(\mathbb{R}^n)$ , and  $\mu \in M^+(\mathbb{R}^n) \cap L^{-1,p'}(\mathbb{R}^n)$ . Then  $u \in L^1(\mathbb{R}^n, \mu)$  (for a quasicontinuous representative of  $u$ ), and*

$$(2.7) \quad \langle \mu, u \rangle = \int_{\mathbb{R}^n} u \, d\mu.$$

We observe that if, under the assumptions of this theorem,  $-\Delta_p u = \mu$ , then it follows (see [MZ97], Theorem 2.34)

$$(2.8) \quad \langle \mu, u \rangle = \int_{\mathbb{R}^n} u \, d\mu = \|u\|_{1,p}^p = \|\mu\|_{-1,p'}^{p'}.$$

For  $0 < \alpha < n$  and  $\sigma \in M^+(\mathbb{R}^n)$ , the Riesz potential of  $\sigma$  is defined by

$$(2.9) \quad \mathbf{I}_\alpha \sigma(x) = \int_0^\infty \frac{\sigma(B(x,r))}{r^{n-\alpha}} \frac{dr}{r} = \frac{1}{n-\alpha} \int_{\mathbb{R}^n} \frac{d\sigma(y)}{|x-y|^{n-\alpha}}, \quad x \in \mathbb{R}^n.$$

For  $1 < p < \infty$  and  $0 < \alpha < \frac{n}{p}$ , the Wolff potential of order  $\alpha$  is defined by

$$\mathbf{W}_{\alpha,p} \sigma(x) = \int_0^\infty \left( \frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}, \quad x \in \mathbb{R}^n.$$

Note that  $\mathbf{W}_{\alpha,2} \sigma = I_{2\alpha} \sigma$  if  $0 < \alpha < \frac{n}{2}$ . In particular,  $\mathbf{W}_{1,2} \sigma = I_2 \sigma$  is the Newtonian potential for  $n \geq 3$ .

We will need the following Wolff's inequality [HW83] (see also [AH96], Sec. 4.5) which gives precise estimates of the energy associated with the Wolff potential:

**Theorem 2.7.** *Suppose  $1 < p < \infty$ ,  $0 < \alpha < \frac{n}{p}$ , and  $\sigma \in M^+(\mathbb{R}^n)$ . Then there exists a constant  $C > 0$  depending only on  $p, \alpha$ , and  $n$  such that*

$$(2.10) \quad \frac{1}{C} \int_{\mathbb{R}^n} (I_\alpha \sigma)^{p'} \, dx \leq \int_{\mathbb{R}^n} \mathbf{W}_{\alpha,p} \sigma \, d\sigma \leq C \int_{\mathbb{R}^n} (I_\alpha \sigma)^{p'} \, dx,$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ .

### 3. EXISTENCE AND MINIMALITY OF FINITE ENERGY SOLUTIONS

In this section, we deduce a necessary and sufficient condition for the existence of a finite energy solution, and construct a minimal solution to (1.1). We will assume that  $1 < p < n$ , since for  $p \geq n$  there are only trivial nonnegative supersolutions on  $\mathbb{R}^n$  (Theorem 2.3; see also [HKM06], Theorem 3.53).

**Lemma 3.1.** *Suppose there exists a nontrivial supersolution  $u \geq 0$ ,  $u \in L_0^{1,p}(\mathbb{R}^n) \cap L_{\text{loc}}^q(\mathbb{R}^n, d\sigma)$  to (1.1). Then*

$$-\Delta_p u \in L^{-1,p'}(\mathbb{R}^n) \cap M^+(\mathbb{R}^n).$$

Moreover,  $u \in L^{1+q}(\mathbb{R}^n, \sigma)$  (for a quasicontinuous representative of  $u$ ), and condition (1.8) holds.

*Proof.* Suppose  $u \in L_0^{1,p}(\mathbb{R}^n) \cap L_{\text{loc}}^q(\mathbb{R}^n, d\sigma)$  is a supersolution to (1.1). Then by Hölder's inequality, for every  $\varphi \in C_0^\infty(\mathbb{R}^n)$ ,

$$|\langle \Delta_p u, \varphi \rangle| = \left| \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \right| \leq \|\nabla u\|_{L^p(\mathbb{R}^n)}^{p-1} \|\nabla \varphi\|_{L^p(\mathbb{R}^n)}.$$

Hence,  $\Delta_p u \in L^{-1,p'}(\mathbb{R}^n)$ . If  $\varphi \geq 0$ , then

$$-\langle \Delta_p u, \varphi \rangle = \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \geq \int_{\mathbb{R}^n} \varphi u^q \, d\sigma \geq 0,$$

and consequently  $-\Delta_p u \in M^+(\mathbb{R}^n)$ .

It follows that  $d\mu = u^q \, d\sigma \in M^+(\mathbb{R}^n) \cap L^{-1,p'}(\mathbb{R}^n)$ . Let  $\{\varphi_j\}$  be a sequence of nonnegative  $C_0^\infty$ -functions such that  $\varphi_j \rightarrow u$  in  $L_0^{1,p}(\mathbb{R}^n)$ . By definition,

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi_j \, dx \geq \langle \mu, \varphi_j \rangle.$$

Hence,

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx = \lim_{j \rightarrow \infty} \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi_j \, dx \geq \lim_{j \rightarrow \infty} \langle \mu, \varphi_j \rangle = \langle \mu, u \rangle.$$

Let us assume as usual that  $u$  coincides with its quasicontinuous representative. Then, applying Theorem 2.6, we deduce

$$\langle \mu, u \rangle = \int_{\mathbb{R}^n} u \, d\mu = \int_{\mathbb{R}^n} u^{1+q} \, d\sigma < \infty.$$

By Theorem 2.3, it follows that if  $u \not\equiv 0$ , then  $u \geq C(\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}$ , and consequently (1.8) holds.  $\square$

**Lemma 3.2.** *For every  $r > 0$ ,*

$$(3.1) \quad \mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^r \, d\sigma)(x) \geq C(\mathbf{W}_{\alpha,p}\sigma(x))^{\frac{r}{p-1}+1}, \quad x \in \mathbb{R}^n,$$

where  $C$  depends only on  $p, q, r, \alpha$ , and  $n$ .

*Proof.* For  $t > 0$ , obviously,

$$\mathbf{W}_{\alpha,p}\sigma(y) = \int_0^t \left( \frac{\sigma(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} + \int_t^\infty \left( \frac{\sigma(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}$$

For  $y \in B(x,t)$ , we have

$$\begin{aligned} \int_t^\infty \left( \frac{\sigma(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} &= \int_{t/2}^\infty \left( \frac{\sigma(B(y,2r))}{(2r)^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &= \left( \frac{1}{2} \right)^{\frac{n-\alpha p}{p-1}} \int_{t/2}^\infty \left( \frac{\sigma(B(y,2s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \geq C_{n,p,\alpha} \int_t^\infty \left( \frac{\sigma(B(y,2s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}, \end{aligned}$$



where  $C_{n,p,\alpha} = \left(\frac{1}{2}\right)^{\frac{n-\alpha p}{p-1}}$ . Since  $s \geq t$  and  $y \in B(x, t)$ , then  $B(y, 2s) \supset B(x, s)$ , which implies

$$(3.2) \quad \mathbf{W}_{\alpha,p}\sigma(y) \geq C_{n,p,\alpha} \int_t^\infty \left( \frac{\sigma(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}.$$

Notice that

$$\mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^r d\sigma)(x) = \int_0^\infty \left( \frac{\int_{B(x,t)} [\mathbf{W}_{\alpha,p}\sigma(y)]^r d\sigma(y)}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

By (3.2), we obtain

$$\begin{aligned} \mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^r d\sigma)(x) &\geq \\ &\geq \int_0^\infty \left( \frac{\int_{B(x,t)} \left[ C_{n,p,\alpha} \int_t^\infty \left( \frac{\sigma(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^r d\sigma(y)}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\geq C_{n,p,\alpha}^{\frac{r}{p-1}} \int_0^\infty \left[ \int_t^\infty \left( \frac{\sigma(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{\frac{r}{p-1}} \left( \frac{\sigma(B(x, t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}. \end{aligned}$$

Integrating by parts, we deduce

$$\mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^r d\sigma)(x) \geq \frac{C_{n,p,\alpha}^{\frac{r}{p-1}}}{\frac{r}{p-1} + 1} \left( \int_0^\infty \left( \frac{\sigma(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right)^{\frac{r}{p-1} + 1}.$$

Thus,

$$\mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^r d\sigma)(x) \geq C_{n,p,\alpha,r} (\mathbf{W}_{1,p}\sigma(x))^{\frac{r}{p-1} + 1}.$$

□

Setting  $r = \frac{q(p-1)}{p-1-q}$  in Lemma 3.2, we deduce

$$(3.3) \quad \mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^{\frac{q(p-1)}{p-1-q}} d\sigma)(x) \geq \kappa (\mathbf{W}_{\alpha,p}\sigma(x))^{\frac{p-1}{p-1-q}},$$

where  $\kappa$  depends only on  $p, q$ , and  $n$ .

Let us define a nonlinear integral operator  $T$  by

$$(3.4) \quad T(f)(x) = \left( \mathbf{W}_{\alpha,p}(f d\sigma) \right)^{p-1}(x), \quad x \in \mathbb{R}^n.$$

**Lemma 3.3.** *Let  $1 < p < \infty$ ,  $0 < \alpha < n$ , and  $0 < q < p - 1$ . Suppose*

$$(3.5) \quad \int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(1+q)(p-1)}{p-1-q}} d\sigma < \infty.$$

*Then  $T$  is a bounded operator from  $L^{\frac{1+q}{q}}(\mathbb{R}^n, d\sigma)$  to  $L^{\frac{1+q}{p-1}}(\mathbb{R}^n, d\sigma)$ .*

*Proof.* Clearly,

$$\|(\mathbf{W}_{\alpha,p}(fd\sigma))^{p-1}\|_{L^{\frac{1+q}{p-1}}(d\sigma)} = \left( \int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}(fd\sigma))^{1+q} d\sigma \right)^{\frac{p-1}{1+q}}.$$

We have

$$\mathbf{W}_{\alpha,p}(fd\sigma)(x) \leq \int_0^\infty \left( \frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{p'-1} M_\sigma f(x)^{p'-1} \frac{dr}{r} = M_\sigma f(x)^{p'-1} \mathbf{W}_{1,p}\sigma(x),$$

where the centered maximal operator  $M_\sigma$  is defined by

$$M_\sigma f(x) = \sup_{r>0} \frac{1}{\sigma(B(x,r))} \int_{B(x,r)} |f| d\sigma, \quad x \in \mathbb{R}^n.$$

It is well known that  $M_\sigma : L^s(\mathbb{R}^n, d\sigma) \rightarrow L^s(\mathbb{R}^n, d\sigma)$  is a bounded operator for all  $s > 1$ . Let  $s = \frac{1+q}{q}$ . Then, using Hölder's inequality with the exponents  $\beta = \frac{p-1}{q} > 1$  and  $\beta' = \frac{p-1}{p-1-q}$ , we estimate,

$$\begin{aligned} \int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}(fd\sigma))^{1+q} d\sigma &\leq \int_{\mathbb{R}^n} (M_\sigma f)^{\frac{1+q}{p-1}} (\mathbf{W}_{\alpha,p}\sigma)^{1+q} d\sigma \\ &\leq \left( \int_{\mathbb{R}^n} (M_\sigma f)^{\frac{1+q}{q}} d\sigma \right)^{\frac{q}{p-1}} \left( \int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(1+q)(p-1)}{p-1-q}} d\sigma \right)^{\frac{p-1-q}{p-1}} \\ &\leq C \left( \int_{\mathbb{R}^n} f^{\frac{1+q}{q}} d\sigma \right)^{\frac{q}{p-1}} \left( \int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(1+q)(p-1)}{p-1-q}} d\sigma \right)^{\frac{p-1-q}{p-1}}. \end{aligned}$$

Thus,

$$\|(\mathbf{W}_{\alpha,p}(fd\sigma))^{p-1}\|_{L^{\frac{1+q}{p-1}}(d\sigma)} \leq c \|f\|_{L^{\frac{1+q}{q}}(d\sigma)}.$$

□

**Remark 3.4.** *It is not difficult to see that actually (3.5) is also necessary for the boundedness of the operator  $T : L^{\frac{1+q}{q}}(\mathbb{R}^n, d\sigma) \rightarrow L^{\frac{1+q}{p-1}}(\mathbb{R}^n, d\sigma)$  (see, for example, [COV06]).*

**Theorem 3.5.** *Let  $1 < p < n$ , and  $0 < q < p-1$ . Suppose that condition (1.8) holds. Then there exists a solution  $u \in L^{1+q}(\mathbb{R}^n, d\sigma)$  to the integral equation (2.5).*

*Proof.* By Lemma 3.3, we have, for all  $f \in L^{\frac{1+q}{q}}(\mathbb{R}^n, d\sigma)$ ,

$$(3.6) \quad \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(fd\sigma))^{1+q} d\sigma \leq C \left( \int_{\mathbb{R}^n} f^{\frac{1+q}{q}} d\sigma \right)^{\frac{q}{p-1}}.$$

Let  $u_0 = c_0 (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}$ , where  $c_0 > 0$  is a small constant to be chosen later on. We construct a sequence of iterations  $u_j$  as follows:

$$(3.7) \quad u_{j+1} = \mathbf{W}_{1,p}(u_j^q d\sigma), \quad j = 0, 1, 2, \dots$$

Applying Lemma 3.2, we have

$$u_1 = \mathbf{W}_{1,p}(u_0^q d\sigma) = c_0^{\frac{q}{p-1}} \mathbf{W}_{1,p}((\mathbf{W}_{1,p}\sigma)^{\frac{q(p-1)}{p-1-q}} d\sigma) \geq c_0^{\frac{q}{p-1}} \kappa (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}},$$

where  $\kappa$  is the constant in (3.1). Choosing  $c_0$  so that  $c_0^{\frac{q}{p-1}} \kappa \geq c_0$ , we obtain  $u_1 \geq u_0$ . By induction, we can show that the sequence  $\{u_j\}$  is nondecreasing. Note that  $u_0 \in L^{1+q}(\mathbb{R}^n, d\sigma)$  by assumption. Suppose that  $u_0, \dots, u_j \in L^{1+q}(\mathbb{R}^n, d\sigma)$ . Then

$$\int_{\mathbb{R}^n} u_{j+1}^{1+q} d\sigma = \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(u_j^q d\sigma))^{1+q} d\sigma.$$

Applying (3.6) with  $f = u_j^q$ , we obtain by induction,

$$(3.8) \quad \int_{\mathbb{R}^n} u_{j+1}^{1+q} d\sigma \leq C \left( \int_{\mathbb{R}^n} u_j^{1+q} d\sigma \right)^{\frac{q}{p-1}} < \infty.$$

Since  $u_j \leq u_{j+1}$ , the preceding inequality yields

$$\int_{\mathbb{R}^n} u_{j+1}^{1+q} d\sigma \leq C \left( \int_{\mathbb{R}^n} u_{j+1}^{1+q} d\sigma \right)^{\frac{q}{p-1}} < \infty.$$

Thus,

$$\left( \int_{\mathbb{R}^n} u_{j+1}^{1+q} d\sigma \right)^{\frac{p-1-q}{p-1}} \leq C < \infty.$$

Using the Monotone Covergence Theorem and passing to the limit as  $j \rightarrow \infty$  in (3.7), we see that there exists  $u = \lim_{j \rightarrow \infty} u_j$ , such that  $u \in L^{1+q}(\mathbb{R}^n, d\sigma)$ , and the integral equation (2.5) holds.  $\square$

**Lemma 3.6.** *Let  $u \in L^{1+q}(\mathbb{R}^n, d\sigma)$  be a nonnegative supersolution to the integral equation (2.5). Then*

$$(3.9) \quad u^q d\sigma \in L^{-1,p'}(\mathbb{R}^n).$$

*Proof.* Let  $d\nu = u^q d\sigma$ . We need to show that, for all  $\varphi \in C_0^\infty(\mathbb{R}^n)$ ,

$$(3.10) \quad \left| \int_{\mathbb{R}^n} \varphi d\nu \right| \leq c \left( \int_{\mathbb{R}^n} |\nabla \varphi|^p dx \right)^{\frac{1}{p}}.$$

It is easy to see that the above inequality is equivalent to

$$(3.11) \quad \left| \int_{\mathbb{R}^n} \mathbf{I}_1 g d\nu \right| \leq c \left( \int_{\mathbb{R}^n} |g|^p dx \right)^{\frac{1}{p}},$$

for all  $g \in L^p(\mathbb{R}^n)$ , where  $\mathbf{I}_1 g$  is the Riesz potential of  $g$  of order 1. By duality, (3.11) is equivalent to

$$(3.12) \quad \int_{\mathbb{R}^n} (\mathbf{I}_1 \nu)^{p'} dx < \infty.$$

Using Wolff's inequality (2.10), we deduce that (3.12) holds if and only if

$$(3.13) \quad \int_{\mathbb{R}^n} \mathbf{W}_{1,p} \nu d\nu < \infty.$$

Notice that since  $u \geq \mathbf{W}_{1,p}(u^q d\sigma)$  and  $u \in L^{1+q}(\mathbb{R}^n, d\sigma)$  then

$$\int_{\mathbb{R}^n} \mathbf{W}_{1,p} \nu d\nu = \int_{\mathbb{R}^n} \mathbf{W}_{1,p}(u^q d\sigma) u^q d\sigma \leq \int_{\mathbb{R}^n} u^{1+q} d\sigma < \infty.$$

Thus, (3.12) holds. This completes the proof of the lemma.  $\square$

We will need a weak comparison principle which goes back to P. Tolksdorf's work on quasilinear equations (see, e.g., [PV08], Lemma 6.9, in the case of renormalized solutions in bounded domains).

**Lemma 3.7.** *Suppose  $\mu, \omega \in M^+(\mathbb{R}^n) \cap L^{-1,p'}(\mathbb{R}^n)$ . Suppose  $u$  and  $v$  are (quasicontinuous) solutions in  $L_0^{1,p}(\mathbb{R}^n)$  of the equations  $-\Delta_p u = \mu$  and  $-\Delta_p v = \omega$ , respectively. If  $\mu \leq \omega$ , then  $u \leq v$  q.e.*

*Proof.* For every  $\varphi \in L_0^{1,p}(\mathbb{R}^n)$ , we have by Theorem 2.6,

$$(3.14) \quad \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \langle \mu, \varphi \rangle = \int_{\mathbb{R}^n} \varphi d\mu,$$

$$(3.15) \quad \int_{\mathbb{R}^n} |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi dx = \langle \omega, \varphi \rangle = \int_{\mathbb{R}^n} \varphi d\omega.$$

Hence,

$$(3.16) \quad \int_{\mathbb{R}^n} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla \varphi dx = \int_{\mathbb{R}^n} \varphi d\mu - \int_{\mathbb{R}^n} \varphi d\omega.$$

Since  $\mu \leq \omega$ , it follows that, for every  $\varphi \in L_0^{1,p}(\mathbb{R}^n)$ ,  $\varphi \geq 0$ , we have

$$(3.17) \quad \int_{\mathbb{R}^n} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla \varphi dx \leq 0.$$

Testing (3.17) with  $\varphi = (u - v)^+ = \max\{u - v, 0\} \in L_0^{1,p}(\mathbb{R}^n)$ , we obtain,

$$I = \int_{\mathbb{R}^n} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla (u - v)^+ dx \leq 0.$$

Let  $A = \{x \in \mathbb{R}^n : u(x) > v(x)\}$ , then

$$I = \int_A (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla (u - v) dx \leq 0.$$

Note that

$$(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla (u - v) \geq 0.$$

Thus,

$$0 \leq \int_A (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla (u - v) dx = \int_A \varphi (d\mu - d\omega) \leq 0.$$

It follows that  $\nabla(u - v) = 0$  a.e. on  $A$ . By Lemma 2.22 in [MZ97], for every  $a > 0$ ,

$$\text{cap}_p \{u - v > a\} \leq \frac{1}{a^p} \int_A |\nabla(u - v)|^p dx = 0.$$

Consequently,  $\text{cap}_p(A) = 0$ , i.e.,  $u \leq v$  q.e.  $\square$

We are now in a position to prove the main theorem of this section.

**Theorem 3.8.** *Let  $1 < p < n$  and  $0 < q < p - 1$ . Let  $\sigma \in M^+(\mathbb{R}^n)$ ,  $\sigma \neq 0$ . Suppose that (1.8) holds. Then there exists a nontrivial solution  $w \in L_0^{1,p}(\mathbb{R}^n) \cap L_{\text{loc}}^q(\mathbb{R}^n, d\sigma)$  to (1.1). Moreover,  $w$  is a minimal solution, i.e.,  $w \leq u$   $d\sigma$ -a.e. (q.e. for quasicontinuous representatives) for any nontrivial solution  $u \in L_0^{1,p}(\mathbb{R}^n) \cap L_{\text{loc}}^q(\mathbb{R}^n, d\sigma)$  to (1.1).*

*Proof.* We first show that there exists a solution  $w \in L_0^{1,p}(\mathbb{R}^n) \cap L_{\text{loc}}^q(\mathbb{R}^n, d\sigma)$  to (1.1). Applying Theorem 3.5, we conclude that there exists a solution  $v \in L^{1+q}(\mathbb{R}^n, d\sigma)$  to the integral equation (2.5). By using a constant multiple  $cv$  in place of  $v$ , we can assume that  $v = K \mathbf{W}_{1,p}(v^q d\sigma)$ , where  $K$  is the constant in Theorem 2.5. Then by Lemma 3.6 and Theorem 2.3,

$$v^q d\sigma \in L^{-1,p'}(\mathbb{R}^n), \quad \text{and} \quad v \geq C K^{\frac{p-1}{p-1-q}} (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}},$$

where  $C$  is the constant in (2.6).

We set

$$w_0 = c_0 (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}, \quad d\omega_0 = w_0^q d\sigma,$$

where  $c_0 > 0$  is a small constant to be determined later. Since

$$w_0 \leq \frac{c_0}{C K^{\frac{p-1}{p-1-q}}} v,$$

it follows that, for  $c_0 \leq C K^{\frac{p-1}{p-1-q}}$ , we have  $w_0 \leq v$ . Hence,

$$w_0 \in L^{1+q}(\mathbb{R}^n, d\sigma), \quad \text{and} \quad \omega_0 \in L^{-1,p'}(\mathbb{R}^n).$$

Then there exists a unique nonnegative solution  $w_1 \in L_0^{1,p}(\mathbb{R}^n)$  to the equation

$$-\Delta_p w_1 = \omega_0, \quad \text{and} \quad \|w_1\|_{1,p}^{p-1} = \|\omega_0\|_{-1,p'}.$$

(See (2.8).) Moreover, by Theorem 2.5,

$$0 \leq w_1 \leq K \mathbf{W}_{1,p}\omega_0 \leq K \mathbf{W}_{1,p}(v^q d\sigma) = v.$$

Consequently, by Lemma 3.6,

$$w_1 \in L^{1+q}(\mathbb{R}^n, d\sigma), \quad \text{and} \quad w_1^q d\sigma \in L^{-1,p'}(\mathbb{R}^n).$$

We deduce, using (3.3),

$$\begin{aligned} w_1 &\geq \frac{1}{K} \mathbf{W}_{1,p}\omega_0 = \frac{c_0^{\frac{q}{p-1}}}{K} \mathbf{W}_{1,p} \left( (\mathbf{W}_{1,p}\sigma)^{\frac{q(p-1)}{p-1-q}} d\sigma \right) \\ &\geq \frac{c_0^{\frac{q}{p-1}} \kappa}{K} (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} = \frac{c_0^{\frac{q}{p-1}-1} \kappa}{K} w_0. \end{aligned}$$

Hence, for  $c_0 \leq (K^{-1} \kappa)^{\frac{p-1}{p-1-q}}$ , we have  $v \geq w_1 \geq w_0$ .

To prove the minimality of  $w$ , we will need  $c_0 \leq C$ , so we pick  $c_0$  so that

$$(3.18) \quad 0 < c_0 \leq \min \left\{ C K^{\frac{p-1}{p-1-q}}, (K^{-1} \kappa)^{\frac{p-1}{p-1-q}}, C \right\}.$$

Let us now construct by induction a sequence  $\{w_j\}_{j \geq 1}$  so that

$$(3.19) \quad \begin{cases} -\Delta_p w_j = \sigma w_{j-1}^q & \text{in } \mathbb{R}^n, \quad w_j \in L_0^{1,p}(\mathbb{R}^n) \cap L^{1+q}(\mathbb{R}^n, d\sigma), \\ 0 \leq w_{j-1} \leq w_j \leq v, & \text{q.e.,} \quad w_{j-1}^q d\sigma \in L^{-1,p'}(\mathbb{R}^n), \end{cases}$$

where  $\sup_j \|w_j\|_{1,p} < \infty$ . We set  $d\omega_j = w_j^q d\sigma$ , so that

$$-\Delta_p w_j = \omega_{j-1}, \quad j = 1, 2, \dots$$

Suppose that  $w_0, w_1, \dots, w_{j-1}$  have been constructed. As in the case  $j = 1$ , we see that, since  $\omega_{j-1} \in L^{-1,p'}(\mathbb{R}^n)$ , there exists a unique  $w_j \in L_0^{1,p}(\mathbb{R}^n)$  such that  $-\Delta_p w_j = \omega_{j-1}$ , and by (2.8),

$$\|w_j\|_{1,p}^p = \|\omega_{j-1}\|_{-1,p'}^{p'} = \int_{\mathbb{R}^n} w_j w_{j-1}^q d\sigma.$$

By Theorem 2.5, we get

$$w_j \leq K \mathbf{W}_{1,p} \omega_{j-1} = K \mathbf{W}_{1,p} (w_{j-1}^q d\sigma).$$

Using the inequality  $w_{j-1} \leq v$ , we see that

$$w_j \leq K \mathbf{W}_{1,p} (v^q d\sigma) = v.$$

Combining these estimates, we obtain

$$\|w_j\|_{1,p}^p = \int_{\mathbb{R}^n} w_j w_{j-1}^q d\sigma \leq \int_{\mathbb{R}^n} v^{1+q} d\sigma < \infty.$$

Consequently,  $\{w_j\}$  is a bounded sequence in  $L_0^{1,p}(\mathbb{R}^n)$ . Notice that  $w_{j-1} \leq w_j$  by the weak comparison principle (Lemma 3.7), since  $\omega_{j-2} \leq \omega_{j-1}$ , for  $j \geq 2$ .

Thus, the sequence (3.19) has been constructed. Letting  $w = \lim_{j \rightarrow \infty} w_j$ , and applying the weak continuity of the  $p$ -Laplace operator (Theorem 2.4), the Monotone Convergence Theorem, and Lemma 1.33 in [HKM06], we deduce the existence of a nontrivial solution  $w \in L_0^{1,p}(\mathbb{R}^n)$  to (1.1).

We now prove the minimality of  $w$ . Suppose  $u \in L_0^{1,p}(\mathbb{R}^n) \cap L_{\text{loc}}^q(\mathbb{R}^n, d\sigma)$  is any nontrivial solution to (1.1). Letting  $d\mu = u^q d\sigma$ , we have  $u \in L^{1+q}(\mathbb{R}^n, d\sigma)$ , and  $\mu \in L^{-1,p'}(\mathbb{R}^n)$  by Lemma 3.1. To show that  $u \geq w$ , notice that by Theorem 2.3,

$$u \geq C (\mathbf{W}_{1,p} \sigma)^{\frac{p-1}{p-1-q}},$$

where  $C$  is the constant in (2.6). By the choice of  $c_0$  in (3.18), we have  $w_0 \leq u$ , so that  $\omega_0 \leq \mu$ . Therefore, by the weak comparison principle  $w_1 \leq u$  q.e. Arguing by induction as above, we see that  $w_{j-1} \leq w_j \leq u$  q.e. for  $j \geq 1$ . It follows that  $\lim_{j \rightarrow \infty} w_j = w \leq u$  q.e., which proves that  $w$  is a minimal solution.  $\square$

By combining Lemma 3.1 and Theorem 3.8 we conclude the proof of the existence part of the Theorem stated in the Introduction. In Sec. 5 below we will establish the uniqueness part using the existence of a minimal solution constructed in Theorem 3.8.

4.  $\mathcal{A}$ -LAPLACE OPERATORS

Let us assume that  $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfies the following structural assumptions:

$$x \rightarrow \mathcal{A}(x, \xi) \quad \text{is measurable for all } \xi \in \mathbb{R}^n,$$

$$\xi \rightarrow \mathcal{A}(x, \xi) \quad \text{is continuous for a.e. } x \in \mathbb{R}^n,$$

and there are constants  $0 < \alpha \leq \beta < \infty$ , such that for a.e.  $x$  in  $\mathbb{R}^n$ , and for all  $\xi$  in  $\mathbb{R}^n$ ,

$$\mathcal{A}(x, \xi) \cdot \xi \geq \alpha |\xi|^p, \quad |\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p-1},$$

$$(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0 \quad \text{if } \xi_1 \neq \xi_2,$$

$$\mathcal{A}(x, \lambda \xi) = \lambda |\lambda|^{p-2} \mathcal{A}(x, \xi), \quad \text{if } \lambda \in \mathbb{R} \setminus \{0\}.$$

Consider the equation

$$(4.1) \quad -\operatorname{div} \mathcal{A}(x, \nabla u) = \mu \quad \text{in } \Omega,$$

where  $\mu \in M^+(\Omega)$ , and  $\Omega \subseteq \mathbb{R}^n$  is an open set. Let us use the decomposition  $\mu = \mu_0 + \mu_s$ , where  $\mu_0$  is absolutely continuous with respect to the  $p$ -capacity and  $\mu_s$  is singular with respect to the  $p$ -capacity. Let  $T_k(s) = \max\{-k, \min\{k, s\}\}$ . We say that  $u$  is a *local renormalized solution* to (4.1) if, for all  $k > 0$ ,  $T_k(u) \in W_{\operatorname{loc}}^{1,p}(\Omega)$ ,  $u \in L_{\operatorname{loc}}^{(p-1)s}$  for  $1 \leq s < \frac{n}{n-p}$ ,  $Du \in L_{\operatorname{loc}}^{(p-1)r}(\Omega)$  for  $1 \leq r < \frac{n}{n-1}$ , and

$$\begin{aligned} & \int_{\Omega} \langle \mathcal{A}(x, Du), Du \rangle h'(u) \phi \, dx + \int_{\Omega} \langle \mathcal{A}(x, Du), \nabla \phi \rangle h(u) \phi \, dx \\ &= \int_{\Omega} h(u) \phi \, d\mu_0 + h(+\infty) \int_{\Omega} \phi \, d\mu_s, \end{aligned}$$

for all  $\phi \in C_0^\infty(\Omega)$  and  $h \in W^{1,\infty}(\mathbb{R})$  such that  $h'$  is compactly supported; here  $h(+\infty) = \lim_{t \rightarrow +\infty} h(t)$ .

In [KKT09], it is shown that every  $\mathcal{A}$ -superharmonic function is locally a renormalized solution, and conversely, every local renormalized solution has an  $\mathcal{A}$ -superharmonic representative. Consequently, we can work either with local renormalized solutions, or equivalently with potential theoretic solutions, or finite energy solutions in the case  $u \in L_0^{1,p}(\Omega)$ . We note that, for finite energy solutions,  $Du$  coincides with the distributional gradient  $\nabla u$ , and  $d\mu = u^q d\sigma$  is absolutely continuous with respect to the  $p$ -capacity as was mentioned above.

It is known that basic facts of potential theory stated in Sec. 2, including Wolff's potential estimates [KM94], and the weak continuity principle [TW02], remain true for the  $\mathcal{A}$ -Laplacian. From the above results it follows that our methods work, with obvious modifications, not only for the  $p$ -Laplace operator, but for the general  $\mathcal{A}$ -Laplace operator  $\operatorname{div} \mathcal{A}(x, \nabla u)$  as well. In particular, the following more general theorem holds.

**Theorem 4.1.** *Under the above assumptions on  $\mathcal{A}(x, \xi)$ , together with the conditions of the Theorem stated in Sec. 1, the equation*

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \sigma u^q$$

*has a solution  $u \in L_0^{1,p}(\mathbb{R}^n) \cap L_{\operatorname{loc}}^q(\mathbb{R}^n, d\sigma)$  if and only if condition (1.8) holds.*

## 5. UNIQUENESS

In this section, we prove the uniqueness of finite energy solutions to (1.1). We employ a convexity argument using some ideas of Kawohl [Kaw00] (see also [BeK02], [BF12]), together with the existence of a minimal solution established above.

**Theorem 5.1.** *Let  $1 < p < \infty$  and let  $0 < q < p - 1$ . Let  $\sigma \in M^+(\mathbb{R}^n)$ . Suppose that there exists a nontrivial solution  $u \in L_0^{1,p}(\mathbb{R}^n) \cap L_{\operatorname{loc}}^q(\mathbb{R}^n, d\sigma)$  to (1.1). Then such a solution is unique.*

*Proof.* Suppose  $u, v$  are nontrivial solutions to (1.1) which lie in  $L_0^{1,p}(\mathbb{R}^n) \cap L_{\operatorname{loc}}^q(\mathbb{R}^n, \sigma)$ . We first show that  $u = v$   $d\sigma$ -a.e. implies that  $u = v$  as elements of  $L_0^{1,p}(\mathbb{R}^n)$ .

Indeed, suppose that  $u = v$   $d\sigma$ -a.e., and set  $d\mu = u^q d\sigma = v^q d\sigma$ , where  $\mu \in M^+(\mathbb{R}^n)$ , and

$$(5.1) \quad -\Delta_p u = -\Delta_p v = \mu.$$

As usual, we assume that  $u, v$  are quasicontinuous representatives (see, e.g., [HKM06], [MZ97]). Then by Lemma 3.1,  $u, v \in L^{1+q}(\mathbb{R}^n, d\sigma)$ , and

$$\int_{\mathbb{R}^n} \mathbf{W}_{1,p} \mu d\mu < +\infty.$$

By Wolff's inequality (2.10), this means that  $\mu \in L^{-1,p'}(\mathbb{R}^n)$ . It is well known ([MZ97], Sec. 2.1.5) that, for such  $\mu$ , a finite energy solution to the equation  $-\Delta_p u = \mu$  is unique. (See also Lemma 3.7 above.) Hence, from (5.1) we deduce  $u = v$  q.e. and as elements of  $L_0^{1,p}(\mathbb{R}^n)$ .

We next show that if  $u \geq v$   $d\sigma$ -a.e. then  $u = v$   $d\sigma$ -a.e. By Theorem 2.3, it follows that  $u(x) > 0, v(x) > 0$ , for all  $x \in \mathbb{R}^n$ . Testing the equations

$$(5.2) \quad \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi dx = \int_{\mathbb{R}^n} u^q \phi d\sigma, \quad \phi \in L_0^{1,p}(\mathbb{R}^n) \cap L_{\operatorname{loc}}^q(\mathbb{R}^n, d\sigma),$$

$$(5.3) \quad \int_{\mathbb{R}^n} |\nabla v|^{p-2} \nabla v \cdot \nabla \psi dx = \int_{\mathbb{R}^n} v^q \psi d\sigma, \quad \psi \in L_0^{1,p}(\mathbb{R}^n) \cap L_{\operatorname{loc}}^q(\mathbb{R}^n, d\sigma),$$

with  $\phi = u, \psi = v$ , respectively, we obtain

$$\int_{\mathbb{R}^n} |\nabla u|^p dx = \int_{\mathbb{R}^n} u^{1+q} d\sigma, \quad \int_{\mathbb{R}^n} |\nabla v|^p dx = \int_{\mathbb{R}^n} v^{1+q} d\sigma.$$

Let

$$\lambda_t(x) = \left( (1-t)v^p(x) + tu^p(x) \right)^{\frac{1}{p}}.$$



Using convexity of the Dirichlet integral  $\int_{\mathbb{R}^n} |\nabla u|^p dx$  in  $u^p$  [Kaw00] (see also the proof of Lemma 2.1 in [BF12]), we estimate, for all  $t \in [0, 1]$ ,

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla \lambda_t(x)|^p dx &\leq (1-t) \int_{\mathbb{R}^n} |\nabla v|^p dx + t \int_{\mathbb{R}^n} |\nabla u|^p dx \\ &= t \left( \int_{\mathbb{R}^n} |\nabla u|^p dx - \int_{\mathbb{R}^n} |\nabla v|^p dx \right) + \int_{\mathbb{R}^n} |\nabla v|^p dx. \end{aligned}$$

Thus,

$$\int_{\mathbb{R}^n} \frac{|\nabla \lambda_t(x)|^p - |\nabla \lambda_0(x)|^p}{t} dx \leq \int_{\mathbb{R}^n} u^{1+q} d\sigma - \int_{\mathbb{R}^n} v^{1+q} d\sigma.$$

Using the inequality

$$|a|^p - |b|^p \geq p|b|^{p-2}b \cdot (a-b), \quad a, b \in \mathbb{R}^n,$$

we deduce

$$|\nabla \lambda_t|^p - |\nabla \lambda_0|^p \geq p|\nabla \lambda_0|^{p-2} \nabla \lambda_0 \cdot (\nabla \lambda_t - \nabla \lambda_0).$$

Notice that  $\lambda_0 = v$ , and consequently, for all  $t \in (0, 1]$ ,

$$(5.4) \quad p \int_{\mathbb{R}^n} |\nabla v|^{p-2} \nabla v \cdot \frac{\nabla(\lambda_t - \lambda_0)}{t} dx \leq \int_{\mathbb{R}^n} u^{1+q} d\sigma - \int_{\mathbb{R}^n} v^{1+q} d\sigma.$$

Testing (5.3) with  $\psi = \lambda_t - \lambda_0 \in L_0^{1,p}(\mathbb{R}^n)$ , we obtain

$$\int_{\mathbb{R}^n} |\nabla v|^{p-2} \nabla v \cdot \nabla(\lambda_t - \lambda_0) dx = \int_{\mathbb{R}^n} v^q (\lambda_t - \lambda_0) d\sigma.$$

Hence, by (5.4), for all  $t \in (0, 1]$ ,

$$(5.5) \quad p \int_{\mathbb{R}^n} v^q \frac{\lambda_t - \lambda_0}{t} d\sigma \leq \int_{\mathbb{R}^n} u^{1+q} d\sigma - \int_{\mathbb{R}^n} v^{1+q} d\sigma.$$

Clearly,  $\lambda_t \geq \lambda_0$ , since  $u \geq v$ . Applying Fatou's lemma, we obtain

$$\int_{\mathbb{R}^n} v^q \frac{u^p - v^p}{v^{p-1}} d\sigma \leq \liminf_{t \rightarrow 0} p \int_{\mathbb{R}^n} v^q \frac{\lambda_t - \lambda_0}{t} d\sigma.$$

Combining this and (5.5) yields

$$\int_{\mathbb{R}^n} \left( \frac{v^q u^p}{v^{p-1}} - v^{1+q} \right) d\sigma \leq \int_{\mathbb{R}^n} u^{1+q} d\sigma - \int_{\mathbb{R}^n} v^{1+q} d\sigma.$$

Therefore, canceling the second terms on both sides, and taking into account that  $u \geq v$   $d\sigma$ -a.e., we arrive at

$$\begin{aligned} 0 &\geq \int_{\mathbb{R}^n} \left( \frac{v^q u^p}{v^{p-1}} - u^{1+q} \right) d\sigma = \int_{\mathbb{R}^n} \frac{v^q u^p - u^{1+q} v^{p-1}}{v^{p-1}} d\sigma \\ &= \int_{\mathbb{R}^n} \frac{v^q u^{1+q} (u^{p-1-q} - v^{p-1-q})}{v^{p-1}} d\sigma \geq 0. \end{aligned}$$

Hence,  $u = v$   $d\sigma$ -a.e.

We now complete the proof of the uniqueness property. Suppose that  $u$  and  $v$  are nontrivial finite energy solutions to (1.1). Then  $\min(u, v) \geq w$   $d\sigma$ -a.e., where  $w$  is the nontrivial minimal solution constructed in Theorem

3.8. Therefore, as was shown above,  $w = u = v$   $d\sigma$ -a.e., and also as elements of  $L_0^{1,p}(\mathbb{R}^n)$ .  $\square$

## REFERENCES

- [ABV10] H. Abdel Hamid and M.-F. Bidaut-Véron, *On the connection between two quasilinear elliptic problems with source terms of order 0 or 1*, Commun. Contemp. Math. **12** (2010), 727–788.
- [AH96] D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren der math. Wissenschaften **314**, Springer, 1996.
- [BeK02] M. Belloni and B. Kawohl, *A direct uniqueness proof for equations involving the  $p$ -Laplace operator*, Manuscr. Math. **109** (2002), 229–231.
- [BO96] L. Boccardo and L. Orsina, *Sublinear elliptic equations in  $L^s$* , Houston Math. J. **20** (1994), 99–114.
- [BO12] L. Boccardo and L. Orsina, *Sublinear elliptic equations with singular potentials*, Adv. Nonlinear Stud. **12** (2012), 187–198.
- [BF12] L. Brasco and G. Franzina, *A note on positive eigenfunctions and hidden convexity*, Arch. Math. **99** (2012), 367–374.
- [BrB79] H. Brezis and F. E. Browder, *A property of Sobolev spaces*, Commun. PDE **44** (1979), 1077–1083.
- [BrK92] H. Brezis and S. Kamin, *Sublinear elliptic equations on  $\mathbb{R}^n$* , Manuscr. Math. **74** (1992), 87–106.
- [BrO86] H. Brezis and L. Oswald, *Remarks on sublinear elliptic equations*, Nonlin. Analysis, Theory, Methods Appl. **10** (1986), 55–64.
- [CV13] D. T. Cao and I. E. Verbitsky, *Existence and pointwise estimates of solutions to subcritical quasilinear elliptic equations*, preprint (2013).
- [COV00] C. Cascante, J. M. Ortega, and I. E. Verbitsky, *Trace inequalities of Sobolev type in the upper triangle case*, Proc. London Math. Soc. **80** (2000), 391–414.
- [COV06] C. Cascante, J. M. Ortega, and I. E. Verbitsky, *On  $L^p - L^q$  trace inequalities*, J. London Math. Soc. **74** (2006), 497–511.
- [HKM06] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Dover Publ., 2006 (unabridged republ. of 1993 edition, Oxford University Press).
- [HW83] L. I. Hedberg and T. Wolff, *Thin sets in nonlinear potential theory*, Ann. Inst. Fourier (Grenoble) **33** (1983), 161–187.
- [JV10] B. J. Jaye and I. E. Verbitsky, *The fundamental solution of nonlinear operators with natural growth terms*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **12** (2013), 93–139.
- [JV12] B. J. Jaye and I. E. Verbitsky, *Local and global behaviour of solutions to nonlinear equations with natural growth terms*, Arch. Rational Mech. Anal. **204** (2012), 627–681.
- [Kaw00] B. Kawohl, *Symmetry results for functions yielding best constants in Sobolev-type inequalities*, Discrete Cont. Dynam. Syst. **6** (2000), 683–690.
- [Kil03] T. Kilpeläinen,  *$p$ -Laplacian type equations involving measures*, Proc. ICM, Vol. III, 167–176, Beijing, 2002.
- [KM92] T. Kilpeläinen and J. Malý, *Degenerate elliptic equations with measure data and nonlinear potentials*, Ann. Scuola Norm. Super. Pisa, Cl. Sci. **19** (1992), 591–613.
- [KM94] T. Kilpeläinen and J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), 137–161.
- [KKT09] T. Kilpeläinen, T. Kuusi and A. Tuhola-Kujanpää, *Superharmonic functions are locally renormalized solutions*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **28** (2011), 775–795.

- [Kr64] M. A. Krasnoselskii, *Positive Solutions of Operator Equations*, P. Noordhoff, Groningen, 1964.
- [KuMi13] T. Kuusi and G. Mingione, *Guide to nonlinear potential estimates*, Bull. Math. Sci., to appear.
- [Lab02] D. Labutin, *Potential estimates for a class of fully nonlinear elliptic equations*, Duke Math. J. **111** (2002), 1–49.
- [MZ97] J. Malý and W. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Math. Surveys Monogr. **51**, Amer. Math. Soc., 1997.
- [Maz11] V. Maz'ya, *Sobolev Spaces, with Applications to Elliptic Partial Differential Equations*, 2nd augmented ed., Grundlehren der mathematischen Wissenschaften **342**, Springer, Berlin, 2011.
- [PV08] N. C. Phuc and I. E. Verbitsky, *Quasilinear and Hessian equations of Lane–Emden type*, Ann. Math. **168** (2008), 859–914.
- [PV09] N. C. Phuc and I. E. Verbitsky, *Singular quasilinear and Hessian equations and inequalities*, J. Funct. Anal. **256** (2009), 1875–1905.
- [PV13] N. C. Phuc and I. E. Verbitsky, *Quasilinear equations with source terms on Carnot groups*, Trans. Amer. Math. Soc. (2013), <http://dx.doi.org/10.1090/S0002-9947-2013-05920-X>
- [TW99] N. S. Trudinger and X.-J. Wang, *Hessian measures II*, Ann. Math. **150** (1999), 579–604.
- [TW02] N. S. Trudinger and X.-J. Wang, *On the weak continuity of elliptic operations and applicaitons to potential theory*, Amer. J. Math. **124** (2002), 369–410.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65211, USA

*E-mail address:* dtcznb@mail.missouri.edu

*E-mail address:* verbitskyi@missouri.edu